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# Inhomogeneous Inverse Differential Realization of Two-Parameter Deformed Quasi-SU(2)$_{q,s}$ Group

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The generators and irreducible representation coherent state of the two-parameter deformed ($q$,$s$-deformed) quasi-SU(2)$_{q,s}$ group are constructed by using the inverse operators of the $q$,$s$-deformed bosonic oscillator, and the inhomogeneous inverse differential realization of the $q$,$s$-deformed quasi-SU(2)$_{q,s}$ group is derived.

## 1. INTRODUCTION

The boson realization approach is very effective for studying the representation of groups, and the boson realization usually can be obtained from the creation and annihilation operators of the bosonic oscillator, such as the Jordan–Schwinger realization, etc. On the other hand, the nature of the inverse operator of the bosonic oscillator has been studied (Dirac, 1966) and new results have been given in recent articles (Fan, 1993, 1994; Fan *et al.*, 1994; Mehta *et al.*, 1992). In consideration of the close relationship between the quantum mechanical quasi-accuracy soluble problem and the inhomogeneous differential realization of the Lie group, we have obtained the inhomogeneous inverse differential realization of the multimode quasi-SU(2) group (Yu *et al.*, 1997). The present paper further studies the inhomogeneous inverse differential realization of the $q$,$s$-deformed quasi-SU(2)$_{q,s}$ group by using the inverse operator of the $q$,$s$-deformed bosonic oscillator.

## 2. INVERSE OF THE $q$,$s$-DEFORMED BOSONIC OSCILLATOR

We introduce two independent $q$,$s$-deformed bosonic oscillators as follows (Jing *et al.*, 1993; Jing, 1993):

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$$a_i^+ a_i = [N_i]_{qs}, \qquad a_i a_i^+ = [N_i + 1]_{qs}, \tag{1}$$
$$[N_i, a_i] = -a_i, \qquad [N_i, a_i^+] = a_i^+ \qquad (i = 1, 2)$$

where the $q$,$s$-deformed bracket is given by $[x]_{qs} = s^{1-x}(q^x - q^{-x})(q - q^{-1})$. It is easy to find that

$$a_i a_i^+ - s^{-1}q\, a_i^+ a_i = (sq)^{-N_i}, \qquad a_i a_i^+ - (sq)^{-1}\, a_i^+ a_i = (s^{-1}q)^{N_i} \tag{2}$$

and we also have

$$a_i|n\rangle = \sqrt{[n]_{qs}}|n - 1\rangle, \qquad a_i^+|n\rangle = \sqrt{[n + 1]_{qs}}|\mathrm{n} + 1\rangle \tag{3}$$

The $q$,$s$-deformed coherent state is constructed as (Yu *et al.*, 1997)

$$|z\rangle_i = e_{qs}^{za_i^+}|0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]_{qs}!}}|n\rangle \tag{4}$$

which is unnormalized. Similar to Fan (F1994), we have

$$\begin{aligned}|n\rangle &= \frac{\sqrt{[n]_{qs}!}}{2\pi} \sum_{m=0}^{\infty} \int_0^{2\pi} d\varphi\, e^{i(m-n)\varphi} \frac{1}{\sqrt{[m]_{qs}!}}|m\rangle \\ &= \frac{\sqrt{[n]_{qs}!}}{2\pi i} \oint_c dz \frac{1}{z^{n+1}}|z\rangle_i \qquad (z = e^{i\varphi})\end{aligned} \tag{5}$$

where the contour encircles the point $z = 0$. In the sense of this contour integration we have $a_i^{-1}|z\rangle_i = z^{-1}|z\rangle_i$. Therefore

$$a_i^{-1}|n\rangle = \frac{\sqrt{[n]_{qs}!}}{2\pi i} \oint_c dz \frac{1}{z^{n+2}} e_{qs}^{za_i^+}|0\rangle = \frac{1}{\sqrt{[n + 1]_{qs}}}|n + 1\rangle \tag{6}$$

which means

$$a_i^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{[n + 1]_{qs}}}|n + 1\rangle\langle n| \qquad (i = 1, 2) \tag{7}$$

On the other hand, we can find by using the same method

$$(a_i^+)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{[n + 1]_{qs}}}|n\rangle\langle n + 1| = (a_i^{-1})^+ \qquad (i = 1, 2) \tag{8}$$

The noncommutative relation between $a_i$ and $a_i^{-1}$ follows:

$$a_i a_i^{-1} = (a_i^+)^{-1} a_i^+ = 1, \qquad a_i^{-1} a_i = a_i^+ (a_i^+)^{-1} = 1 - |0\rangle\langle 0| \tag{9}$$

where the vacuum projection operator $|0\rangle\langle 0|$ can be obtained from $a_i|0\rangle\langle 0| = 0$, namely (Yu *et al.*, 1996)

$$|0\rangle\langle 0| = \sum_{n=0}^{\infty} \frac{(-1)^n (a_i^+)^n (s^{-1}q)^{n(N_i+n-1)} a_i^n}{[n]_{qs}!} \tag{10}$$

## 3. INHOMOGENEOUS INVERSE DIFFERENTIAL REALIZATION OF THE *q,s*-DEFORMED QUASI-SU(2)$_{q,s}$ GROUP

Consider the following combinations of the inverse of $\{a_1^+, a_1, N_1\}$ and $\{a_2^+, a_2, N_2\}$:

$$J_+^{-1} = (a_1^+)^{-1} a_2^{-1}, \qquad J_-^{-1} = (a_2^+)^{-1} a_1^{-1} \tag{11}$$

$$J_0^{-1} = \frac{1}{2} \{(N_1 + 1)^{-1} N_2^{-1} - N_1^{-1}(N_2 + 1)^{-1}\} \tag{12}$$

where the notations are defined as

$$N_1^{-1} = a_1^{-1}(a_1^+)^{-1}, \qquad (N_1 + 1)^{-1} = (a_1^+)^{-1} a_1^{-1} \tag{13}$$

$$N_2^{-1} = a_2^{-1}(a_2^+)^{-1}, \qquad (N_2 + 1)^{-1} = (a_2^+)^{-1} a_2^{-1} \tag{14}$$

which satisfy the relations

$$N_i^{-1}|n\rangle = \frac{1}{[n]_{qs}} |n\rangle, \qquad (N_i + 1)^{-1}|n\rangle = \frac{1}{[n + 1]_{qs}} |n\rangle \qquad (i = 1, 2) \tag{15}$$

Therefore we get

$$s^{-1} J_+^{-1} J_-^{-1} - s J_-^{-1} J_+^{-1} = s^{-2J_0^{-1}} [2J_0^{-1}] \tag{16}$$

Equation (16) denotes that a nonclosed $q,s$-deformed SU(2)$_{q,s}$ group is constructed; we call it a quasi-SU(2)$_{q,s}$ group.

The unitary irreducible representations $|j, m\rangle$ of the $q,s$-deformed quasi-SU(2)$_{q,s}$ group are

$$|j, m\rangle = |j + m\rangle \otimes |j - m\rangle \qquad (-j \leq m \leq j) \tag{17}$$

These irreducible representations are finite and depend on a single quantum number $j = 0, 1/2, 1, \ldots$. The actions of the $q,s$-deformed quasi-SU(2)$_{q,s}$ group inverse generators on the elements of the irreducible representation (17) are given by

$$J_+^{-1}|j, m\rangle = \frac{1}{\sqrt{[j+m]_{qs}[j-m+1]_{qs^{-1}}}}|j, m-1\rangle \tag{18}$$

$$J_-^{-1}\,|j, m\rangle = \frac{1}{\sqrt{[j+m+1]_{qs}[j-m]_{qs^{-1}}}}|j, m+1\rangle \tag{19}$$

$$J_0^{-1}|j, m\rangle = \frac{[j+m]_{qs}[j-m+1]_{qs^{-1}} - [j+m+1]_{qs}[j-m]_{qs^{-1}}}{2[j+m+1]_{qs}[j+m]_{qs}[j-m]_{qs^{-1}}[j-m+1]_{qs^{-1}}}|j, m\rangle \tag{20}$$

The coherent state of the irreducible representation for the $q$,$s$-deformed quasi-SU(2) $_{q,s}$ group is written as

$$|jZ) = e^{zJ_-^{-1}}|j, -j\rangle = \sum_{m=-j}^{j} \frac{1}{(j+m)!} \sqrt{\frac{[j-m]_{qs^{-1}}!}{[j+m]_{qs}!\ [2j]_{qs^{-1}}!}}\ z^{j+m}|j, m\rangle \tag{21}$$

where the normalization coefficient is

$$C_j(|z|^2) = \sum_{m=-j}^{j} \frac{[j-m]_{qs^{-1}}!}{\{(j+m)!\}^2[j+m]_{qs}!\ [2j]_{qs^{-1}}!}\ (|z|^2)^{j+m} \tag{22}$$

The method we used here is based on Yu *et al.* (1995a,b). In order to construct the completeness relation of the quantum state $|jz)$, we define $P(j + m, Z)$ to be an observable probability of $|j, m\rangle$ in state $|jz)$:

$$P(j+m, Z) = |\langle j, m|jz)|^2 = \frac{[j-m]_{qs^{-1}}!}{(j+m)!\ [j+m]_{qs}!\ [2j]_{qs^{-1}}!}\ (|z|^2)^{j+m} \tag{23}$$

Setting $P(j + m) = \int P(j + m, Z)\ dz^2$ and letting $\rho$ represent the density matrix of state $|j, m\rangle$, we have

$$\rho = \sum_{m=-j}^{j} P(j+m)|j, m\rangle\langle j, m| \tag{24}$$

Therefore the completeness relation of the quantum state $|jz)$ can be written as

$$\frac{1}{\pi}\rho^{-1}\int \frac{|jz)(jz|}{C_j(|z|^2)}\,dz^2 = 1 \tag{25}$$

We now define the Bargmann representation of the bases $|j, m\rangle$ for the irreducible representations as follows:

$$f_{jm}(z) = (j\bar{z}|j, m\rangle = \frac{1}{(j+m)!}\sqrt{\frac{[j-m]_{qs^{-1}}!}{[j+m]_{qs}!\,[2j]_{qs^{-1}}!}} \tag{26}$$

Furthermore, we also define a state vector in the space of the irreducible representation

$$|\psi\rangle = \sum_{m=-j}^{j} C_m|j, m\rangle \tag{27}$$

We get

$$\begin{aligned}(j\bar{z}|J_-^{-1}|\psi\rangle &= \sum_{m=-j}^{j} C_m\,(j\bar{z}|J_-^{-1}|j, m\rangle \\ &= \sum_{m=-j}^{j} \frac{C_m}{(j+m+1)!\,[j+m+1]_{qs}[j-m]_{qs^{-1}}} \\ &\quad\times \sqrt{\frac{[j-m]_{qs^{-1}}!}{[j+m]_{qs}!\,[2j]_{qs^{-1}}!}}\,Z^{j+m+1}\end{aligned} \tag{28}$$

On the other hand, we have, by using the inverse derivative formula,

$$\begin{aligned}&\frac{1}{[2j - zd/dz]_{qs^{-1}}(1/z)(zd/dz)[zd/dz]_{qs}}\,(j\bar{z}|\psi\rangle \\ &\quad= \sum_{m=-j}^{j} \frac{C_m}{(j+m+1)!\,[j+m+1]_{qs}[j-m]_{qs^{-1}}} \\ &\quad\quad\times \sqrt{\frac{[j-m]_{qs^{-1}}!}{[j+m]_{qs}!\,[2j]_{qs^{-1}}!}}\,Z^{j+m+1}\end{aligned} \tag{29}$$

From Eqs. (28) and (29), we have the inhomogeneous inverse differential realization of the operator $J_-^{-1}$:

$$B_j(J_-^{-1}) = \frac{1}{[2j - zd/dz]_{qs^{-1}}(1/z)(zd/dz)[zd/dz]_{qs}} \tag{30}$$

Similarly, we also have

$$
\begin{aligned}
(j\bar{z}|J_+^{-1}|\psi\rangle
&= \sum_{m=-j}^{j} C_m (j\bar{z}|J_+^{-1}|j, m\rangle \\
&= \sum_{m=-j}^{j} \frac{C_m}{(j+m-1)!} \sqrt{\frac{[j-m]_{qs^{-1}}!}{[j+m]_{qs}!\,[2j]_{qs^{-1}}!}}\, Z^{j+m-1}
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\frac{d}{dz}(j\bar{z}|\psi\rangle
&= \sum_{m=-j}^{j} \frac{C_m}{(j+m-1)!} \sqrt{\frac{[j-m]_{qs^{-1}}!}{[j+m]_{qs}!\,[2j]_{qs^{-1}}!}}\, Z^{j+m-1}
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
(j\bar{z}|J_0^{-1}|\psi\rangle
&= \sum_{m=-j}^{j} C_m (j\bar{z}|J_0^{-1}|j, m\rangle \\
&= \frac{1}{2}\sum_{m=-j}^{j} \frac{C_m([j+m]_{qs}[j-m+1]_{qs^{-1}} - [j+m+1]_{qs}[j-m]_{qs^{-1}})}{(j+m)!\,[j+m]_{qs}[j+m+1]_{qs}[j-m]_{qs^{-1}}[j-m+1]_{qs^{-1}}} \\
&\quad \times \sqrt{\frac{[j-m]_{qs^{-1}}!}{[j+m]_{qs}!\,[2j]_{qs^{-1}}!}}\, Z^{j+m}
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
&\{[zd/dz]_{qs}[2j - zd/dz + 1]_{qs^{-1}} - [zd/dz + 1]_{qs}[2j - zd/dz]_{qs^{-1}}\} \\
&\quad \times \frac{1}{[2j - zd/dz]_{qs^{-1}}(1/z)[2j - zd/dz]_{qs^{-1}}[d/dz]_{qs}(z^2)[d/dz]_{qs}}(j\bar{z}|\psi\rangle \\
&= \frac{1}{2}\sum_{m=-j}^{j} \frac{C_m([j+m]_{qs}[j-m+1]_{qs^{-1}} - [j+m+1]_{qs}[j-m]_{qs^{-1}})}{(j+m)!\,[j+m]_{qs}[j+m+1]_{qs}[j-m]_{qs^{-1}}[j-m+1]_{qs^{-1}}} \\
&\quad \times \sqrt{\frac{[j-m]_{qs^{-1}}!}{[j+m]_{qs}!\,[2j]_{qs^{-1}}!}}\, Z^{j+m}
\end{aligned} \tag{34}
$$

Therefore we can get the inhomogeneous differential realizations of the operators $J_+^{-1}$ and $J_0^{-1}$ in the Bargmann space,

$$B_j(J_+^{-1}) = d/dz \tag{35}$$

$$\begin{aligned} B_j(J_0^{-1}) = {} & \{[zd/dz]_{qs}[2j - zd/dz + 1]_{qs^{-1}} \\ & - [zd/dz + 1]_{qs}[2j - zd/dz]_{qs^{-1}}\} \\ & \times \frac{1}{[2j - zd/dz]_{qs^{-1}}(1/z)[2j - zd/dz]_{qs^{-1}}[d/dz]_{qs}(z^2)[d/dz]_{qs}} \end{aligned} \tag{36}$$

## 4. CONCLUSIONS

From the above discussion, we conclude that Eqs. (30), (35), and (36) are the inhomogeneous inverse differential realizations of the $q,s$-deformed quasi-SU(2) $_{a,s}$ group.

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